# UNIT - I

## Linear model

### Operations Research (OR)

Operation Research is "a scientific approach to decision making, which seeks to determine how best to design and operate a system, under conditions requiring the allocation of scarce resources".

- Effective problem solving & Decision making
- Extensive applications in Engineering, business and Public systems
- Used extensively in manufacturing and Service industries in decision making.
- origin during World War - II

### Linear Programming:

- George B Dantzig around 1947
- Simplex method was published in 1949 by Dantzig.

### Formulation of L.P.P:

① Decision Variables

② Objective function

③ Constraints

④ Non negativity restriction

① Consider a small manufacturer making two products A and B. Two resources $R_1$ and $R_2$ are required to make these products. Each unit of product A requires 1 unit of $R_1$ and 3 units of $R_2$. Each unit of product B requires 1 unit of $R_1$ and 2 units of $R_2$. The manufacturer has 5 units of $R_1$ and 12 units of $R_2$ available. The manufacturer also makes a profit of Rs. 6 per unit of product A sold and Rs. 5 per unit of product B sold. Formulate the L.P.P.

Solution:

Let,

X - Number of units of A to be produced

Y - Number of units of B to be produced

Decision Variables

| | $R_1$ | $R_2$ | |
|---|---|---|---|
| A | 1 | 3 | Rs. 6 |
| B | 1 | 2 | Rs. 5 |
| | 5 | 12 | |

Maximize

Profit $Z = 6X + 5Y$ — objective function

Subject to

$$X + Y \leq 5$$
$$3X + 2Y \leq 12$$

— Constraints

$$X, Y \geq 0$$

— Non-negativity restriction

① Identifying the decision variables

② Writing the objective function

③ Writing the constraints

④ Writing the non-negativity restrictions

Constraints:

$\boxed{\text{Should not having negative value at R.H.S}}$

1. Less than or equal to $\leq$
2. Equal to $=$
3. Greater than or equal to $\geq$

Graphical method: [Less than 3 variables] [only 2 variables]

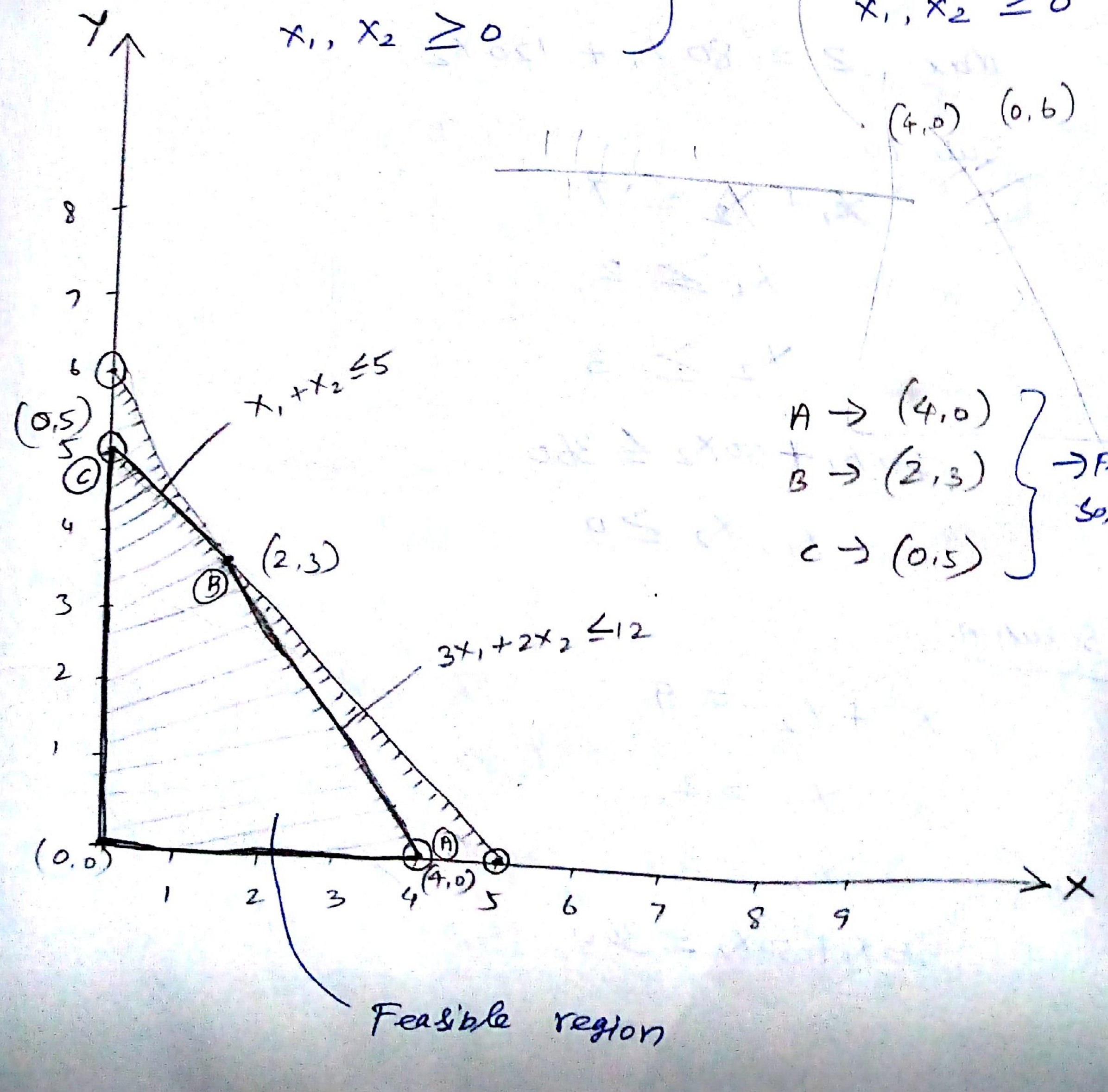
1. Maximize $Z = 6X_1 + 5X_2$

Sub to

$$\left.\begin{array}{l} X_1 + X_2 \leq 5 \\ 3X_1 + 2X_2 \leq 12 \\ X_1, X_2 \geq 0 \end{array}\right\} \Rightarrow \begin{array}{l} X_1 + X_2 = 5 \\ 3X_1 + 2X_2 = 12 \\ X_1, X_2 \geq 0 \end{array}$$

(5,0) (0,5)

(4,0) (0,6)




$$\left.\begin{array}{l} A \to (4,0) \\ B \to (2,3) \\ C \to (0,5) \end{array}\right\} \to \text{Feasible Solution}$$

| Point | $(x_1, x_2)$ | $Z = 6x_1 + 5x_2$ |
| --- | --- | --- |
| A | (4,0) | 24 |
| B | (2,3) | 27 |
| C | (0,5) | 25 |

Optimal Solution

$\therefore$ $\boxed{x_1 = 2 \quad x_2 = 3 \quad z = 27}$

② Max $z = 80x_1 + 120x_2$

Sub to

$$x_1 + x_2 \le 9$$

$$x_1 \ge 2$$

$$x_2 \ge 3$$

$$20x_1 + 50x_2 \le 360$$

$$x_1, x_2 \ge 0$$

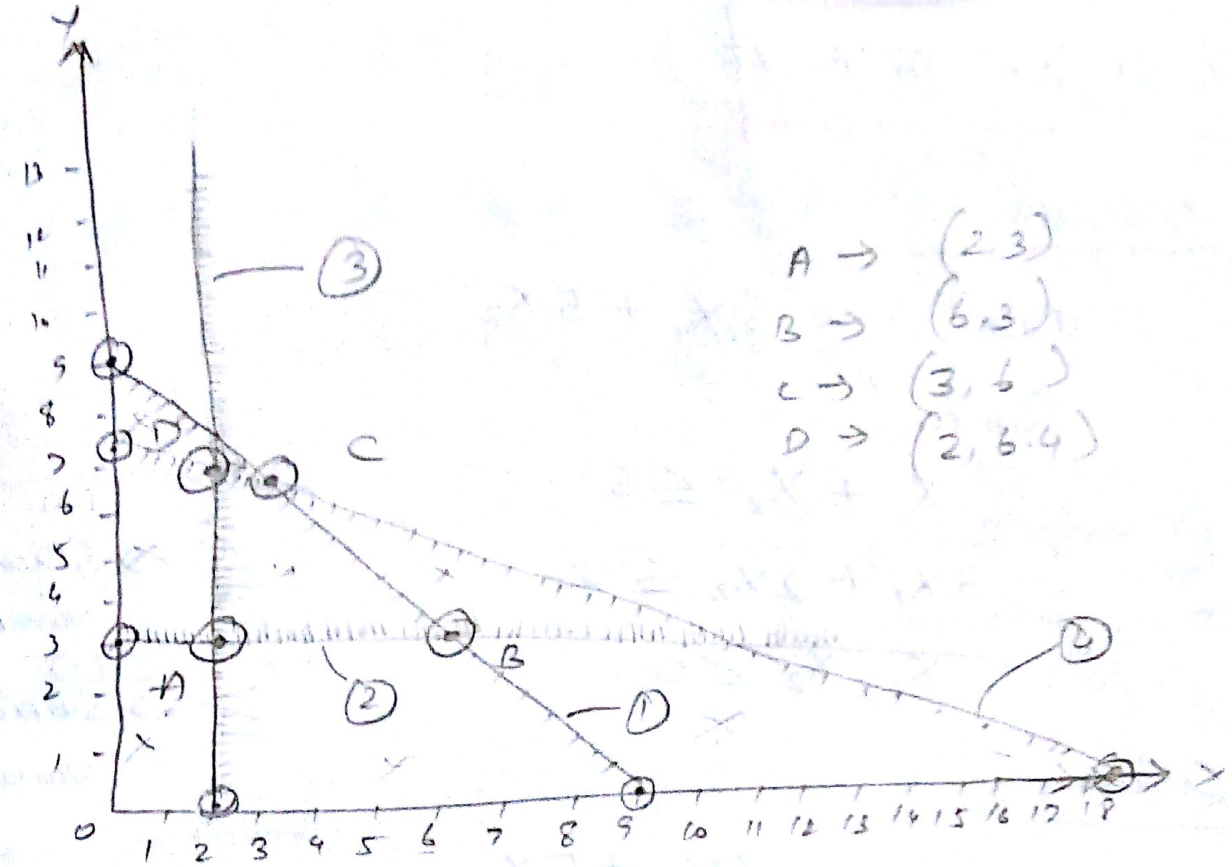
Solution:

$x_1 + x_2 = 9$ (0,9) (9,0)

$x_1 = 2$ (2,0)

$x_2 = 3$ (0,3)

$20x_1 + 50x_2 = 360$ (18,0) (0,7.2)




| Point | $(x_1, x_2)$ | $Z = 80x_1 + 120x_2$ |
|---|---|---|
| A | (2,3) | 520 |
| B | (6,3) | 840 |
| C | (3,6) | 960 |
| D | (2,6.4) | 928 |

$\therefore$

$$\boxed{\begin{aligned} x_1 &= 3 \\ x_2 &= 6 \\ Z &= 960 \end{aligned}}$$

## Graphical method.

(2) Max $Z = 4X_1 + 3X_2$

Sub to.

$$2X_1 + X_2 \leq 1000$$
$$X_1 + X_2 \leq 800$$
$$X_1 \leq 400$$
$$X_2 \leq 700$$
$$X_1, X_2 \geq 0$$

Solution:

Max $Z = 4X_1 + 3X_2$

Sub to

$$2X_1 + X_2 = 1000 \rightarrow (1)$$
$$X_1 + X_2 = 800 \rightarrow (2)$$
$$X_1 = 400 \rightarrow (3)$$
$$X_2 = 700 \rightarrow (4)$$
$$X_1, X_2 \geq 0$$

(1) if $X_1 = 0 \Rightarrow X_2 = 1000$ ; if $X_2 = 0 \Rightarrow X_1 = 500$

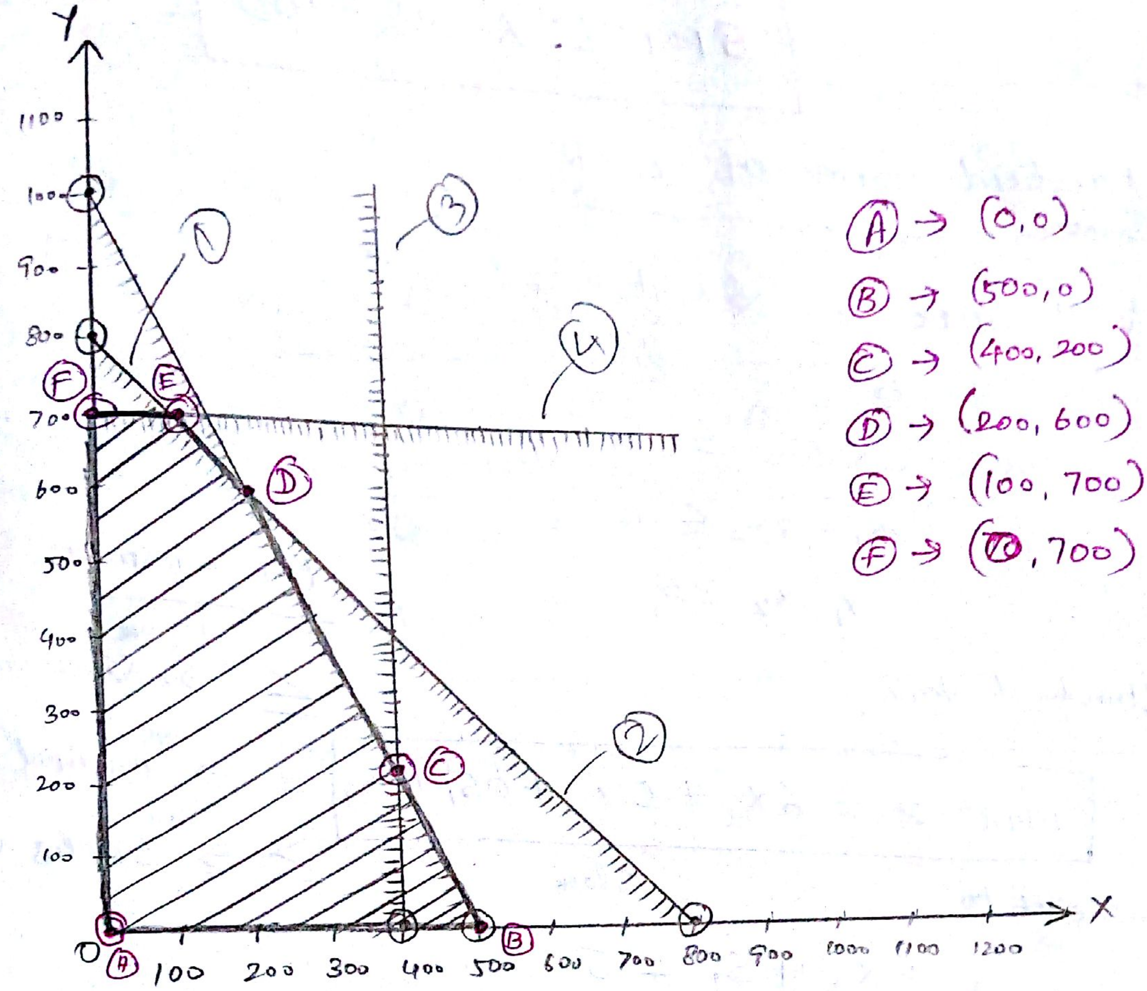
(500, 1000)

(2) if $X_1 = 0 \Rightarrow X_2 = 800$ ; if $X_2 = 0 \Rightarrow X_1 = 800$

(800, 800)

(3) $X_1 = 400$ (400, 0)

(4) $X_2 = 700$ (0, 700)




| Point | $(x_1, x_2)$ | $Z = 4x_1 + 3x_2$ |
|---|---|---|
| A | (0,0) | 0 |
| B | (500,0) | 2000 |
| C | (400, 200) | 2200 |
| D | (200, 600) | 2600 |
| E | (100, 700) | 2500 |
| F | (0,700) | 2100 |

Answer: (Result)

$$\boxed{\begin{aligned} x_1 &= 200 \\ x_2 &= 600 \\ Z &= 2600 \end{aligned}}$$

# Simplex method

## Algebric method:

$$\text{Max } Z = 6X_1 + 5X_2$$

Sub to

$$X_1 + X_2 \leq 5$$
$$3X_1 + 2X_2 \leq 12$$
$$X_1, X_2 \geq 0$$

$\leq \rightarrow$ Slack variable (+)

$\geq \rightarrow$ Surplus Variable (−)

## Solution:

$$\text{Max } Z = 6X_1 + 5X_2$$

Sub to

$$X_1 + X_2 + S_1 = 5$$
$$3X_1 + 2X_2 + S_2 = 12$$
$$X_1, X_2, S_1, S_2 \geq 0$$

## Iteration-1:

$X_1, X_2 \rightarrow$ Non basic variables

$S_1, S_2 \rightarrow$ Basic variables

$$S_1 = 5 - X_1 - X_2$$
$$S_2 = 12 - 3X_1 - 2X_2$$
$$Z = 6X_1 + 5X_2$$

limit of $X_1 = (5, 4)$

## Iteration-2:

$$\therefore 3X_1 = 12 - 2X_2 - S_2$$

$$\Rightarrow X_1 = 4 - \frac{2}{3}X_2 - \frac{1}{3}S_2$$

$$\Rightarrow S_1 = 5 - \left[4 - \frac{2}{3}X_2 - \frac{1}{3}S_2\right] - X_2$$

$$= 5 - 4 + \frac{2}{3}X_2 + \frac{1}{3}S_2 - X_2$$

$\therefore\ S_1 = 1 - \frac{1}{3}x_2 + \frac{1}{3}S_2$

$\therefore\ Z = 6\left[4 - \frac{2}{3}x_2 - \frac{1}{3}S_2\right] + 5x_2$

$= 24 - 4x_2 - 2S_2 + 5x_2$

$$\boxed{Z = 24 + x_2 - 2S_2}$$

Going to increase $x_2$ value

limit of $x_2 = (6, 3)$

Iteration - 3:

$\frac{1}{3}x_2 = 1 + \frac{1}{3}S_2 - S_1$

$\Rightarrow x_2 = 3 + S_2 - 3S_1$

$\therefore\ x_1 = 4 - \frac{2}{3}(3 - 3S_1 + S_2) - \frac{1}{3}S_2$

$= 4 - 2 + 2S_1 - \frac{2}{3}S_2 - \frac{1}{3}S_2$

$= 2 + 2S_1 - S_2$

$\therefore\ Z = 24 + (3 + S_2 - 3S_1) - 2S_2$

$= 27 - S_2 - 3S_1 \rightarrow$ Reaches optimal

$\therefore$ $\boxed{\begin{array}{l} Z = 27 \\ x_1 = 2 \\ x_2 = 3 \end{array}}$

$S_1$ & $S_2$ are 0

## Tabular form:

① max $Z = 6x_1 + 5x_2$

Sub to

$x_1 + x_2 \leq 5$

$3x_1 + 2x_2 \leq 12$

$x_1, x_2 \geq 0$

$x_3$ & $x_4$ are slack variables

## Solution:

Max $Z = 6x_1 + 5x_2 + 0x_3 + 0x_4$

Sub to

$x_1 + x_2 + x_3 + 0x_4 = 5$

$3x_1 + 2x_2 + 0x_3 + x_4 = 12$

$x_1, x_2, x_3, x_4 \geq 0$

## Iteration - 1:

Co-efficients

| | $C_j$ | 6 | 5 | 0 | 0 | | Min |
|---|---|---|---|---|---|---|---|
| | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | R.H.S | $\theta$ |
| 0 | $x_3$ | 1 | 1 | 1 | 0 | 5 | 5 |
| 0 | $x_4$ | 3 | 2 | 0 | 1 | 12 | 4 → |
| | $Z_j$ | 0 ↑ | 0 | 0 | 0 | 0 | |
| | $C_j - Z_j$ | 6 | 5 | 0 | 0 | 0 | |

corresponding element

key column

key element

Basic variables

Dot product $(0\times1)+(0\times3)$

largest $C_j - Z_j$ is key column which enters

$\theta =$ R.H.S / key column

$\therefore 5/1$

min $\theta$ will leaves is key row

Iteration - 2: ($x_1$ enters $x_4$ leaves)

| | 6 | 5 | 0 | 0 | | min |
|---|---|---|---|---|---|---|
| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | R.H.S | $\theta$ |
| 0 $x_3$ | 0 | 1/3 | 1 | -1/3 | 1 | 3 |
| 6 $x_1$ | 1 | 2/3 | 0 | 1/3 | 4 | 6 |
| $z_j$ | 6 | 4 | 0 | 2 | 24 | |
| $c_j - z_j$ | 0 | 1↑ | 0 | -2 | | |

Step ①: $x_1 = \dfrac{\text{key row}}{\text{key element}}$

Step ②: $x_3$ = without affecting the identity matrix by subtracting ~~row 2 from~~ $x_3$ by new $x_1$

(or)

= Old element − $\dfrac{\text{Corresponding element in key column} \times \text{Corresponding element in key row}}{\text{key element}}$

Iteration - 3: ($x_3$ leaves $x_2$ enters)

| | 6 | 5 | 0 | 0 | | |
|---|---|---|---|---|---|---|
| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | R.H.S | $\theta$ |
| 5 $x_2$ | 0 | 1 | 3 | -1 | 3 | |
| 6 $x_1$ | 1 | 0 | -2 | 1 | 2 | |
| $z_j$ | 6 | 5 | 3 | 1 | 27 | |
| $c_j - z_j$ | 0 | 0 | -3 | -1 | | |

Solution reaches optimality $c_j - z_j \leq 0$

$\therefore$ $\boxed{x_1 = 2 \quad x_2 = 3 \quad Z = 27}$

# SIMPLEX METHOD

## Standard form of LPP:

Problem: Max $Z = 6x_1 + 5x_2$

Sub to.

$x_1 + x_2 \leq 5$

$3x_1 + 2x_2 \leq 12$

$x_1, x_2 \geq 0$

For constrains!

$\leq \rightarrow$ use Slack variable

$= \rightarrow$ use Artificial variable

$\geq \rightarrow$ use Surplus variable

### Standard form:

$$\boxed{\text{Max } Z = 6x_1 + 5x_2 + 0S_1 + 0S_2}$$

Sub to,

$$x_1 + x_2 \underbrace{+ S_1}_{\text{Slack}} = 5$$

$$3x_1 + 2x_2 \underbrace{+ S_2}_{\text{Slack}} = 12$$

$x_1, x_2, S_1, S_2 \geq 0$

(n) No. of variables = 4

(m) No. of constraints = 2

$\therefore \boxed{n - m = 2} \rightarrow$ No. of non basic variably

### matrix form:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{bmatrix}_{2\times4} \begin{bmatrix} x_1 \\ x_2 \\ S_1 \\ S_2 \end{bmatrix}_{1\times4} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}_{2\times1}$$

### Variables:

* Basic variables $[S_1, S_2]$ $\rightarrow$ Forms an identity matrix
* Non-basic variables $[x_1, x_2]$ $\rightarrow$ zero Valued
* Non basic variables $\rightarrow$ Initial basic feasible Solution (IBFS)

### Final optimal Solution:

① For maximization $\rightarrow$ All $C_j - Z_j \leq 0$

② For minimization $\rightarrow$ All $C_j - Z_j \geq 0$

Iteration - 1 : (IBFS)

| $C_j$ | 6 | 5 | 0 | 0 | | min |
|---|---|---|---|---|---|---|
| $C_B$ | $X_1$ | $X_2$ | $S_1$ | $S_2$ | R.H.S | $\theta$ [R.H.S / key column] |
| 0 $S_1$ | 1 | 1 | 1 | 0 | 5 | 5 |
| 0 $S_2$ | 3 | 2 | 0 | 1 | 12 | 4 → |
| $Z_j$ | 0 | 0 | 0 | 0 | 0 | |
| max $C_j - Z_j$ | 6 | ↑5 | 0 | 0 | | |

Iteration - 2 : [$X_1$ enters and $S_2$ leaves]

| $C_j$ | 6 | 5 | 0 | 0 | | min |
|---|---|---|---|---|---|---|
| $C_B$ | $X_1$ | $X_2$ | $S_1$ | $S_2$ | R.H.S | $\theta$ |
| 0 $S_1$ | 0 | 1/3 | 1 | -1/3 | 1 | 3 |
| 6 $X_1$ | 1 | 2/3 | 0 | 1/3 | 4 | 6 |
| $Z_j$ | 6 | 4 | 0 | 2 | 24 | |
| max $C_j - Z_j$ | 0 | 1 | 0 | -2 | | |

Iteration - 3 : [$S_1$ leaves & $X_2$ enters]

| $C_j$ | 6 | 5 | 0 | 0 | | min. |
|---|---|---|---|---|---|---|
| $C_B$ | $X_1$ | $X_2$ | $S_1$ | $S_2$ | R.H.S | $\theta$ |
| 5 $X_2$ | 0 | 1 | 3 | -1 | 3 | |
| 6 $X_1$ | 1 | 0 | -2 | 1 | 2 | |
| $Z_j$ | 6 | 5 | 3 | 1 | 27 | |
| max $C_j - Z_j$ | 0 | 0 | -3 | -1 | | |

All $C_j - Z_j \leq 0$ ∴ The solution reaches optimality.

∴ $X_1 = 2$, $X_2 = 3$, $Z = 27$

② Maximize $Z = 3X_1 + 2X_2 + 5X_3$

Sub to,

$X_1 + 4X_2 \leq 420$

$3X_1 + 2X_3 \leq 460$

$X_1 + 2X_2 + X_3 \leq 430$

$X_1, X_2, X_3 \geq 0$

Solution:

Standard form

max. $Z = 3X_1 + 2X_2 + 5X_3 + 0S_1 + 0S_2 + 0S_3$

Sub to

$1X_1 + 4X_2 + 0X_3 + S_1 = 420$

$3X_1 + 0X_2 + 2X_3 + S_2 = 460$

$1X_1 + 2X_2 + 1X_3 + S_3 = 430$

$X_1, X_2, X_3, S_1, S_2, S_3 \geq 0$

Iteration-1 (IBFS):

| $C_j$ | 3 | 2 | 5 | 0 | 0 | 0 | | |
|---|---|---|---|---|---|---|---|---|
| $C_B$ | $X_1$ | $X_2$ | $X_3$ | $S_1$ | $S_2$ | $S_3$ | R.H.S | min $\theta$ |
| 0 $S_1$ | 1 | 4 | 0 | 1 | 0 | 0 | 420 | $\infty$ |
| 0 $S_2$ | 3 | 0 | 2 | 0 | 1 | 0 | 460 | 230 |
| 0 $S_3$ | 1 | 2 | 1 | 0 | 0 | 1 | 430 | 430 |
| $Z_j$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| max $C_j - Z_j$ | 3 | 2 | 5 | 0 | 0 | 0 | | |

Iteration - 2 : [$S_2$ leaves and $x_3$ enters]

| $C_j$ | 3 | 2 | 5 | 0 | 0 | 0 | R.H.S | min |
|---|---|---|---|---|---|---|---|---|
| $C_B$ | $x_1$ | $x_2$ | $x_3$ | $S_1$ | $S_2$ | $S_3$ | | $\theta$ |
| 0 $S_1$ | 1 | 4 | 0 | 1 | 0 | 0 | 420 | 105 |
| 5 $x_3$ | 3/2 | 0 | 1 | 0 | 1/2 | 0 | 230 | $\infty$ |
| 0 $S_3$ | -1/2 | 2 | 0 | 0 | -1/2 | 1 | 200 | 100 |
| $Z_j$ | 15/2 | 0 | 5 | 0 | 5/2 | 0 | 1150 | |
| max $C_j - Z_j$ | -9/2 | 2 | 0 | 0 | -5/2 | 0 | | |

Iteration - 3 : [$S_3$ leaves and $x_2$ enters]

| $C_j$ | 3 | 2 | 5 | 0 | 0 | 0 | R.H.S | min |
|---|---|---|---|---|---|---|---|---|
| $C_B$ | $x_1$ | $x_2$ | $x_3$ | $S_1$ | $S_2$ | $S_3$ | | $\theta$ |
| 0 $S_1$ | 2 | 0 | 0 | 1 | 1 | -2 | 20 | |
| 5 $x_3$ | 3/2 | 0 | 1 | 0 | 1/2 | 0 | 230 | |
| 2 $x_2$ | -1 | 1 | 0 | 0 | -1/4 | 1/2 | 100 | |
| $Z_j$ | 11/2 | 2 | 5 | 0 | 2 | 1 | 1350 | |
| max $C_j - Z_j$ | -5/2 | 0 | 0 | 0 | -2 | -1 | | |

All $C_j - Z_j \leq 0$. The solution reaches optimality.

$\therefore$
$$\boxed{\begin{aligned} x_1 &= 0 \\ x_2 &= 100 \\ x_3 &= 230 \\ z &= 1350 \end{aligned}}$$

# BIG-M METHOD

* Used for $=$ (or) $\geq$ type constraints
* Purpose of introducing artificial variables is just to obtain an initial basic feasible solution (IBFS).
* Maximization → Large Penalty → ($-M$)
* Minimization → Large Penalty → ($+M$)

① Use Big-M method to solve

$$\text{minimize } Z = 4x_1 + 3x_2$$

Sub to
$$2x_1 + x_2 \geq 10$$
$$-3x_1 + 2x_2 \leq 6$$
$$x_1 + x_2 \geq 6$$
$$x_1, x_2 \geq 0$$

Solution:

Standard form
$$\text{Min. } Z = 4x_1 + 3x_2 + 0S_1 + 0S_2 + 0S_3 + MA_1 + MA_2$$

Sub to
$$2x_1 + x_2 - S_1 + A_1 = 10$$
$$-3x_1 + 2x_2 + S_2 = 6$$
$$x_1 + x_2 - S_3 + A_2 = 6$$
$$x_1, x_2, S_1, S_2, S_3, A_1, A_2 \geq 0$$

Iteration-1:

| $C_B$ | $C_j$ | 4 | 3 | 0 | 0 | 0 | $+M$ | $+M$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | $A_1$ | $A_2$ | R.H.S | min $\theta$ |
| $+M$ | $A_1$ | 2 | 1 | $-1$ | 0 | 0 | 1 | 0 | 10 | 5 |
| 0 | $S_2$ | $-3$ | 2 | 0 | 1 | 0 | 0 | 0 | 6 | $-2$ |
| $+M$ | $A_2$ | 1 | 1 | 0 | 0 | $-1$ | 0 | 1 | 6 | 6 |
| | $Z_j$ | $3M$ | $2M$ | $-M$ | 0 | $-M$ | $M$ | $M$ | $16M$ | |
| | Min $C_j - Z_j$ | $4-3M$ | $3-2M$ | $0+M$ | 0 | $M$ | 0 | 0 | | |

Iteration - 2 : [$A_1$ leaves and $x_1$ enters]

| $C_j$ | 4 | 3 | 0 | 0 | 0 | +M | +M | | |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | $A_1$ | $A_2$ | R.H.S | min $\theta$ |
| 4 $x_1$ | 1 | $1/2$ | $-1/2$ | 0 | 0 | $1/2$ | 0 | 5 | 10 |
| 0 $S_2$ | 0 | $7/2$ | $-3/2$ | 1 | 0 | $3/2$ | 0 | 21 | 6 |
| +M $A_2$ | 0 | $1/2$ | $1/2$ | 0 | $-1$ | $-1/2$ | 1 | 1 | 2 |
| $Z_j$ | 4 | $2+M/2$ | $-2+M/2$ | 0 | $-M$ | $2-M/2$ | M | 20+M | |
| min $C_j - Z_j$ | 0 | $1-M/2$ | $2-M/2$ | 0 | M | $M+\frac{M}{2}$ | 0 | | |

Iteration - 3 : [$A_2$ leaves and $x_2$ enters]

| $C_j$ | 4 | 3 | 0 | 0 | 0 | M | M | | |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | $A_1$ | $A_2$ | R.H.S | min $\theta$ |
| 4 $x_1$ | 1 | 0 | $-1$ | 0 | 1 | - | - | 4 | |
| 0 $S_2$ | 0 | 0 | $-5$ | 1 | 7 | - | - | 14 | |
| 3 $x_2$ | 0 | 1 | 1 | 0 | $-2$ | - | - | 2 | |
| $Z_j$ | 4 | 3 | $-1$ | 0 | $-2$ | - | - | 22 | |
| min $C_j - Z_j$ | 0 | 0 | 1 | 0 | 2 | - | - | | |

Since all $C_j - Z_j \geq 0$. ∴ The solution reaches optimal.

$$\therefore \boxed{\begin{aligned} Z &= 22 \\ x_1 &= 4 \\ x_2 &= 2 \end{aligned}}$$

# TWO PHASE METHOD

* Used for $=$ (or), $\geq$ type constraints.

* Phase - ① $\rightarrow$ Max $z^* = (-1)A_1 + (-1)A_2 + \ldots$

  Min $z^* = (1)A_1 + (1)A_2 + \ldots$

  └ If artificial variable appears in final table at non-zero level then stop the procedure.

  └ If not go to phase - II

* Phase - ② $\rightarrow$ Remove all artificial from objective function

  Max $z^* = aX_1 + bX_2 + 0S_1 + 0S_2$

① Use two phase method to solve

Minimize $Z = 4X_1 + 3X_2$

sub to,

$2X_1 + X_2 \geq 10$

$-3X_1 + 2X_2 \leq 6$

$X_1 + X_2 \geq 6$

$X_1, X_2 \geq 0$

Solution:

Standard form of LPP

min. $z = 4X_1 + 3X_2 \pm 0S_1 + 0S_2 + 0S_3 + MA_1 + MA_2$

Sub to,

$2X_1 + X_2 - S_1 + A_1 = 10$

$-3X_1 + 2X_2 + S_2 = 6$

$X_1 + X_2 - S_3 + A_2 = 6$

$X_1, X_2, S_1, S_2, S_3, A_1, A_2 \geq 0$